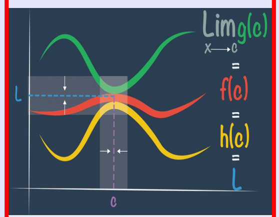
Calculus I

Lecture 8



Feb 19-8:47 AM

Class Quiz 8

Evaluate $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ for $f(x) = x^2 - 2x$.

$$= \lim_{h\to 0} \frac{(x+h)^2 - 2(x+h) - (x^2-2x)}{h} = \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{2x} - 2h - \cancel{x^2} + \cancel{2x}}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2 - 2h}{h} = \lim_{h\to 0} \frac{\cancel{h}(2x + h - 2)}{\cancel{h}}$$

$$= \lim_{h\to 0} (2x + h - 2) = \boxed{2x - 2}$$

Mar 10-8:45 AM

$f(x) = x^2 - 2x$

Y-Int $(0,0)$

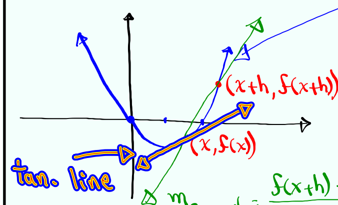
X-Int $(\ , 0)$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0 \quad , x = 2$

X-Ints $(0,0)$, $(2,0)$



$$m_{\substack{\text{secant}\\ \text{line}}} = \frac{f(x+h) - f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}$$

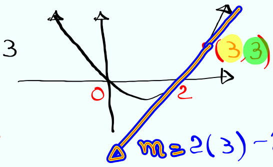
$$m_{\text{tan. line}} = \lim_{h \to 0} m_{\text{secant line}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$m_{\text{tan. line}} = 2x - 2$ (see the quiz)

Suppose $x = 3$

$f(3) = 3^2 - 2(3)$

$= 9 - 6 = 3$



$m = 2(3) - 2 = 4$

tan. line at the point where $x = 3$

$y - y_1 = m(x - x_1)$

$y - 3 = 4(x - 3)$

$y = 4x - 12 + 3$

$\boxed{y = 4x - 9}$

Mar 10-9:03 AM

Find $\boxed{\text{points}}$ on the graph of $f(x) = x^3 - 3x$ $\boxed{(1,-2), (-1,2)}$

where we have horizontal tan. line.

$f(x) = x^3 - 3x \longrightarrow$ odd function

Y-Int $(0,0)$

X-Int $(0,0), (\pm\sqrt{3}, 0)$

$f(x) = 0 \rightarrow x^3 - 3x = 0$

$x(x^2 - 3) = 0$

$x = 0$

$x^2 - 3 = 0$

$x^2 = 3$

$x = \pm\sqrt{3}$

$f(-x) = (-x)^3 - 3(-x)$

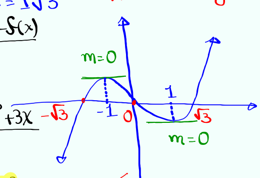
$= -x^3 + 3x$

$= -(x^3 - 3x)$

$= -f(x)$

Symmetric w/t the origin

$$m_{\text{tan. line}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



$$= \lim_{h \to 0} \frac{(x+h)^3 - 3(x+h) - x^3 + 3x}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 \cancel{-3x} - 3h \cancel{-x^3} \cancel{+3x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 3)}{\cancel{h}} = \lim_{h \to 0} (3x^2 + 3xh + h^2 - 3)$$

$= 3x^2 - 3$

Since horizontal line have 0 slope

$3x^2 - 3 = 0$

$3(x^2 - 1) = 0$

$3(x+1)(x-1) = 0$

$x = -1, \ x = 1$

Mar 10-9:12 AM

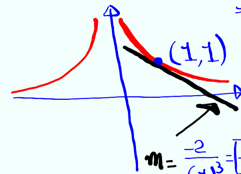
find equation of the tan. line to the graph of $f(x)=\frac{1}{x^2}$ at $x=1$. $f(1)=\frac{1}{1^2}=1$

$f(x)=\frac{1}{x^2}$ $\quad x\neq 0,\ y>0$

$f(x)$ is an even function

$f(-x)=\frac{1}{(-x)^2}=\frac{1}{x^2}=f(x)$

Symmetric w/t the y-axis.



$m=\frac{-2}{(1)^3}=\boxed{-2}$

$$m=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$

LCD $=(x+h)^2\cdot x^2$

$$=\lim_{h\to 0}\frac{x^2-(x+h)^2}{h(x+h)^2\cdot x^2}=\lim_{h\to 0}\frac{\cancel{x^2}-\cancel{x^2}-2xh-h^2}{h(x+h)^2\cdot x^2}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(-2x-h)}{\cancel{h}(x+h)^2\cdot x^2}=\lim_{h\to 0}\frac{-2x-\underset{\to 0}{h}}{(x+\underset{\to 0}{h})^2\cdot x^2}$$

$$=\frac{-2x}{x^2\cdot x^2}=\frac{-2x}{x^4}=\boxed{\frac{-2}{x^3}}$$

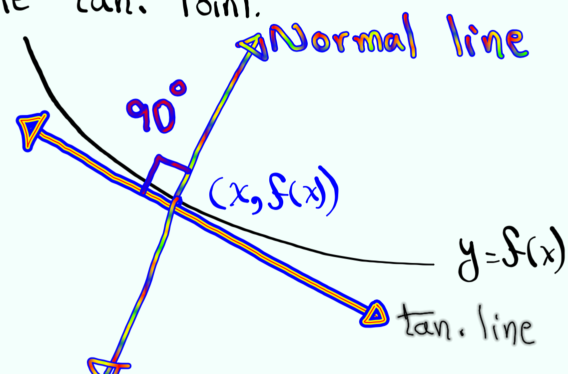
$y-y_1=m(x-x_1)$

$y-1=-2(x-1)$ $\longrightarrow$ $\boxed{y=-2x+3}$

Mar 10-9:25 AM

Normal line

It is perpendicular to the tangent line at the tan. Point.

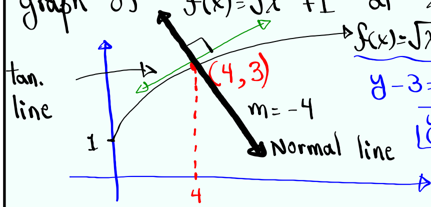


$L_1\perp L_2 \Leftrightarrow m_1\cdot m_2=-1$

$$m_{\substack{\text{Normal}\\ \text{Line}}}=\frac{-1}{m_{\substack{\text{tan.}\\ \text{Line}}}}$$

Mar 10-9:40 AM

Find $\boxed{\text{slope of the normal line}}$ to the graph of $f(x)=\sqrt{x}+1$ at $x=4$.



$y-3=-4(x-4)$

$\boxed{y=-4x+19}$ Normal line

$$m_{\text{tan. line}}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}\cancel{+1}-\sqrt{x}\cancel{-1}}{h}$$

$$=\lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{\cancel{x}+\cancel{h}-\cancel{x}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

at $x=4$ $\quad m_{\text{tan. line}}=\frac{1}{2\sqrt{4}}=\frac{1}{4}$

$$m_{\text{Normal line}}=\frac{-1}{m_{\text{tan. line}}}=\frac{-1}{\frac{1}{4}}=\boxed{-4}$$

Mar 10-9:44 AM

Evaluate $\lim_{x\to 2}\frac{\sin(x-2)}{x^3-8}=\frac{\sin(2-2)}{2^3-8}=\frac{\sin 0}{8-8}=\frac{0}{0}$ I.F.

$$\lim_{x\to 2}\frac{\sin(x-2)}{x^3-8}=\lim_{x\to 2}\frac{\sin(x-2)}{(x-2)(x^2+2x+4)}$$

$$=\lim_{x\to 2}\left[\frac{\sin(x-2)}{x-2}\cdot\frac{1}{x^2+2x+4}\right]$$

$$=\underbrace{\boxed{\lim_{x\to 2}\frac{\sin(x-2)}{x-2}}}_{\to 1}\cdot\lim_{x\to 2}\frac{1}{x^2+2x+4}$$

$$=1\cdot\frac{1}{2^2+2(2)+4}=\boxed{\frac{1}{12}}$$

Mar 10-10:10 AM

Prove $\lim_{x \to 3} (x^2 + 5x - 3) = 21$

1) $f(x) = x^2 + 5x - 3$ $\quad L = 21 \checkmark$ $\quad a = 3$

2) Verify the limit. $\lim_{x \to 3} (x^2+5x-3) = 3^2 + 5(3) - 3$
$= 9 + 15 - 3$
$= 21 \checkmark$

3) $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^2 + 5x - 3 - 21| < \varepsilon$ " $|x-3| < \delta$

$|x^2 + 5x - 24| < \varepsilon$ " $|x-3| < \delta$

$|(x+8)(x-3)| < \varepsilon$

$\underbrace{|x+8|}_{\text{Bound}}\underbrace{|x-3|}_{\text{Keep}} < \varepsilon$

If $|x+8| < C$, then $|x+8||x-3| < C|x-3| < \varepsilon$

$|x-3| < \frac{\varepsilon}{C}$

If $\delta \le 1$, then $|x-3| < 1$

$-1 < x - 3 < 1$
$+3 \quad +3 \quad +3$
$2 < x < 4$
$+8 \quad +8 \quad +8$
$10 < x + 8 < 12 \Rightarrow |x+8| < 12$

Pick
$\delta = \min\left\{1, \frac{\varepsilon}{12}\right\}$

Mar 10-10:15 AM

Prove $\lim_{x \to 2} x^3 = 8$

$f(x) = x^3$ $\quad L = 8 \checkmark$ $\quad a = 2$

$\lim_{x \to 2} x^3 = 2^3 = 8 \checkmark$

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^3 - 8| < \varepsilon$ " $|x-2| < \delta$

$|(x-2)(x^2+2x+4)| < \varepsilon$

$|x-2||x^2+2x+4| < \varepsilon$
Keep $\quad$ Bound

If $\delta \le 1$, then $|x-2| < 1$

$-1 < x - 2 < 1$
$1 < x < 3$

If $x < 3$, then $x^2 < 9$
$2x < 6$

So
$x^2 + 2x + 4 < 9 + 6 + 4$
$x^2 + 2x + 4 < 19$
$|x^2 + 2x + 4| < 19$

Pick
$\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}$

Mar 10-10:24 AM

Prove $\lim\limits_{x\to 1} \dfrac{1}{2x-1} = 1$ $\quad x \ne \frac{1}{2}$

$2x-1 \ne 0 \qquad 2x \ne 1$

$f(x) = \dfrac{1}{2x-1}$ $\qquad L = 1$ ✓ $\qquad a = 1$

verify the limit $\lim\limits_{x\to 1} \dfrac{1}{2x-1} = \dfrac{1}{2(1)-1} = \dfrac{1}{2-1} = \dfrac{1}{1} = 1$

$|f(x) - L| < \varepsilon$ whenever $|x-a| < \delta$

$\left| \dfrac{1}{2x-1} - 1 \right| < \varepsilon \qquad |x-1| < \delta$

$\left| \dfrac{1}{2x-1} - \dfrac{2x-1}{2x-1} \right| < \varepsilon$

$\left| \dfrac{1 - 2x + 1}{2x-1} \right| < \varepsilon$

$\left| \dfrac{-2x+2}{2x-1} \right| < \varepsilon$

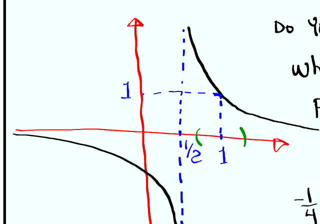
$\left| \dfrac{-2(x-1)}{2x-1} \right| < \varepsilon$

$\underbrace{\dfrac{2}{|2x-1|}}_{\text{Bound}} \underbrace{|x-1|}_{\text{Keep}} < \varepsilon$

If $\dfrac{2}{|2x-1|} < C$, then

$|x-1| < \dfrac{\varepsilon}{C}$

Mar 10-10:31 AM



Do you want $\delta \le 1$? NO

what about $\delta \le \frac{1}{2}$? NO

Pick $\delta \le \frac{1}{4}$

$|x-1| < \frac{1}{4}$

$-\frac{1}{4} < x-1 < \frac{1}{4}$

$-.25 < x-1 < .25$

$+1$

$.75 < x < 1.25$

$1.5 < 2x < 2.5$

-1

$.5 < 2x-1 < 1.5$

$\frac{1}{2} < 2x-1 < \frac{3}{2}$

$2 > \dfrac{1}{2x-1} > \dfrac{2}{3}$

$\frac{2}{3} < \dfrac{1}{2x-1} < 2$

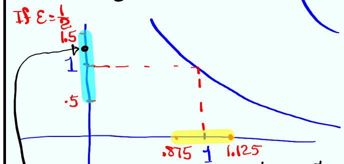
$\frac{4}{3} < \dfrac{2}{2x-1} < 4$

$\dfrac{2}{|2x-1|} < 4$

$C = 4$

$\delta = \min\left\{ \frac{1}{4}, \frac{\varepsilon}{4} \right\}$

If $\varepsilon = \frac{1}{2}$



$f(.9) = \dfrac{1}{2(.9)-1} = \dfrac{1}{1.8-1} = \dfrac{1}{.8} = 1.25$

$\delta = \min\left\{ \frac{1}{4}, \frac{\frac{1}{2}}{4} \right\}$

$= \min\left\{ \frac{1}{4}, \frac{1}{8} \right\}$

$= \frac{1}{8} = .125$

Mar 10-10:40 AM

Prove $\lim_{x \to -5} (x^2 + 5x) = 0$

$f(x) = x^2 + 5x$ $\quad L = 0 \checkmark$ $\quad a = -5$

$\lim_{x \to -5} (x^2 + 5x) = (-5)^2 + 5(-5) = 25 - 25 = 0 \checkmark$

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^2 + 5x - 0| < \varepsilon$ " $|x - (-5)| < \delta$

$|x^2 + 5x| < \varepsilon$ " $|x + 5| < \delta$

$|x(x + 5)| < \varepsilon$

$\underset{\text{Bound}}{|x|}\ \underset{\text{Keep}}{|x + 5|} < \varepsilon$

If $\delta \le 1$, then $|x + 5| < 1$

$-1 < x + 5 < 1$

$-5 \quad -5 \quad -5$

$-6 < x < -4 < 6$

$-6 < x < 6$

$\delta = \min\left\{1, \frac{\varepsilon}{6}\right\}$ $\quad |x| < 6$

Mar 10-10:53 AM

Evaluate $\lim_{x \to 4} \frac{\sqrt{x+5} - 3}{\sqrt{x} - 2} = \frac{\sqrt{4+5} - 3}{\sqrt{4} - 2} = \frac{\sqrt{9} - 3}{\sqrt{4} - 2}$

$= \frac{3-3}{2-2}$

$= \frac{0}{0}$

I.F.

$$= \lim_{x \to 4} \frac{(\sqrt{x+5} - 3)(\sqrt{x} + 2)(\sqrt{x+5} + 3)}{(\sqrt{x} - 2)(\sqrt{x} + 2)(\sqrt{x+5} + 3)}$$

$$= \lim_{x \to 4} \frac{\cancel{(x + 5 - 9)}(\sqrt{x} + 2)}{\cancel{(x - 4)}(\sqrt{x+5} + 3)} = \lim_{x \to 4} \frac{\sqrt{x} + 2}{\sqrt{x+5} + 3}$$

$$= \frac{\sqrt{4} + 2}{\sqrt{4+5} + 3} = \frac{4}{6} = \boxed{\frac{2}{3}}$$

Mar 10-11:01 AM

Evaluate $\lim_{x \to 1} f(x)$ if

$$1 - \sqrt[5]{x} \leq f(x) \leq \sqrt[3]{x} - 1$$

$$\lim_{x \to 1} (1 - \sqrt[5]{x}) = 0, \quad \lim_{x \to 1} (\sqrt[3]{x} - 1) = 0$$

So $\lim_{x \to 1} f(x) = 0$ by S.T.

Mar 10-11:08 AM

Evaluate $\lim_{x \to 1} \frac{x^3 + 5x^2 - 7x + 1}{x^2 - 1} = \frac{1^3 + 5(1)^2 - 7(1) + 1}{1^2 - 1}$

$$= \frac{0}{0}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x^2 + 6x - 1)}{(x+1)\cancel{(x-1)}} = \lim_{x \to 1} \frac{x^2 + 6x - 1}{x + 1}$$

1	1	5	−7	1
		1	6	−1
	1	6	−1	0

$$= \frac{1^2 + 6(1) - 1}{1 + 1}$$

$$= \boxed{3}$$

Mar 10-11:11 AM